that $y_2 = 0$; then a cycloid with the respective inclinations 0 and π at the end points, *i.e.*, a single arch with cusps at the end points, gives a smaller area than any portion of an arch with intermediate inclinations at these same end points. It thus follows that the area given by a single complete arch of a cycloid is smaller than that given by any other curve whose initial inclination is not smaller than 0 and whose terminal inclination is not greater than π.

The method employed above applies to integrals of the form (2) with any number of conditions such as (3) in which other functions of θ as well as the trigonometric may appear in the integrand. Certain light restrictions are to be placed upon such functions.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REMARKS BY J. P. BALLANTINE, Pennsylvania State College.

In remarks by the editor on this question [*1920*, 302], attention was called to the related equation

$$\int_\alpha^\beta f(x)dx = \frac{\beta - \alpha}{6}\left[f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta)\right], \tag{1}$$

where both α and β are allowed to vary; and it was pointed out that if $f(x)$ is a solution of this equation possessing a fifth derivative, then $f(x)$ must be a polynomial of degree $\leqq 3$. It will be proved here that the same conclusion follows from the assumption that $f(x)$ is merely continuous.

I. *Let x_0, x_2, x_4, x_6 be four equally spaced points, and x_1, x_3, x_5 the points of bisection of the three equal intervals formed. Then any integrable function which satisfies* (1) *and vanishes at x_0, x_2, x_4, x_6 vanishes also at x_1, x_3, x_5.*

Let $x_2 - x_0 = 2a$. We find from (1)

$$\int_{x_0}^{x_4} f(x)dx = 0, \qquad \int_{x_2}^{x_6} f(x)dx = 0.$$

Hence, with the notation

$$\int_{x_2}^{x_4} f(x)dx = G, \tag{2}$$

we find

$$\int_{x_0}^{x_2} f(x)dx = -G, \tag{3}$$

$$\int_{x_4}^{x_6} f(x)dx = -G; \tag{4}$$

and therefore, by addition,

$$\int_{x_0}^{x_6} f(x)dx = -G + G - G = -G. \tag{5}$$

By (1) and (2),

$$G = \tfrac{4}{3}af(x_3);$$

by (1) and (5),

$$-G = 4af(x_3);$$

hence

$$G = 0, \qquad f(x_3) = 0.$$

Finally, by (1) and (3),

$$\tfrac{4}{3}af(x_1) = -G = 0,$$

and by (1) and (4),

$$\tfrac{4}{3}af(x_5) = -G = 0;$$

so that

$$f(x_1) = f(x_3) = f(x_5) = 0,$$

as stated.

II. *If an integrable function $f(x)$ satisfying* (1) *vanishes at four equally spaced points, it vanishes also at all points obtainable by repeated bisection of the intervals between the points.*

This is proved by repeated application of I.

III. *If a continuous function $f(x)$ satisfying* (1) *vanishes at four equally spaced points, it vanishes at every point of the interval bounded by the extreme points.*

Since the set of vanishing points obtained in II is everywhere dense, the identical vanishing of $f(x)$ follows at once from the added hypothesis of continuity.

IV. *Any continuous function $f(x)$ satisfying* (1) *is a polynomial of degree* $\leqq 3$.

Choose at random four equally spaced points x_0, x_2, x_4, x_6. It is possible to find $\varphi(x) = A + Bx + Cx^2 + Dx^3$, so that $\varphi(x_0) = f(x_0)$, $\varphi(x_2) = f(x_2)$, $\varphi(x_4) = f(x_4)$, $\varphi(x_6) = f(x_6)$; for on writing out these conditions, we have for the determination of A, B, C, D four linear algebraic equations whose determinant does not vanish. Then the function $f_1(x) = f(x) - \varphi(x)$ satisfies (1) and vanishes at x_0, x_2, x_4, x_6. Therefore, by III, $f_1(x)$ vanishes, $x_0 \leqq x \leqq x_6$. That is, $f(x)$ is a polynomial of degree $\leqq 3$ in the interval from x_0 to x_6. By choosing x_0, x_2, x_4, x_6 properly, any real value of x may be included. It is also easily seen that widening the interval cannot alter the values of A, B, C, D; for, if such alteration were possible, we should have two different polynomials identically equal in the smaller interval, which is impossible.

II. Remarks by Louis Weisner, New York City.

The solution of this problem, published in the Monthly, *1920*, 301–2, is based on the assumption that $f(x)$ is analytic; this assumption leading to the conclusion that "$f(x)$ can be at worst a cubic polynomial." The following solution, based upon a different assumption, leads to a more general result. Assume that

$$f(x) = c_0 + c_1x^{r_1} + c_2x^{r_2} + \cdots + c_nx^{r_n} + \cdots,$$

in which the r's are to be found, if they exist, the only restriction placed upon them being that they may not be negative. Then $f(0) = c_0$.

Substituting for $f(x)$ in the proposed equation, transposing and collecting terms, we find that c_0 disappears and that

$$\sum_{n=1}^{n=\infty} \left[c_n \left(\frac{1}{6} \cdot \frac{4}{2^{r_n}} + \frac{1}{6} - \frac{1}{r_n + 1} \right) h^{r_n+1} \right] = 0.$$

This equation will be satisfied if we choose the r's so that they are the roots of the auxiliary equation

$$\frac{1}{6} \cdot \frac{4}{2^r} + \frac{1}{6} - \frac{1}{r+1} = 0,$$

or of the equation

$$(r+1)(2^{-r+2} + 1) = 6.$$

It is seen by inspection that there are no negative or fractional roots of the auxiliary equation; the only positive integral roots are $r = 1$, $r = 2$, $r = 3$. However, the auxiliary equation has in fact an infinite number of roots, real or complex.

Consequently, we have

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \sum_{n=4}^{n=\infty} c_nx^{r_n},$$

in which the r's are the incommensurable or complex roots of the auxiliary equation. The c's are independent arbitrary constants, since no assumption was made regarding them.

III. Remarks by the Editor.

Mr. Ballantine's result, dealing not with the given equation but with the similar one in which both limits of integration are allowed to vary, is of considerable interest, as it not only shows that the hypothesis of continuity alone suffices to restrict solutions to cubic polynomials, but also effects the proof in a very simple and elementary way. It will be remembered that in virtue of the previous remarks of Professor Bennett and the editor [*1920*, 462], a similar result is not obtainable for the equation as stated in the question, even with the hypothesis of a continuous first derivative, and that it remains undecided what number of derivatives between one and six will suffice.

Mr. Weisner, seeking solutions other than cubic polynomials, is led to a result practically equivalent to that stated by the editor [*1920*, 463]. This result was passing through the press at the time of receipt of Mr. Weisner's manuscript. It may be observed that the editor's statement shows that in Mr. Weisner's notation, when $n > 3$, r_n is necessarily of the form $2 + \beta i$, where β is a solution of a certain transcendental equation.

DISCUSSIONS.

The two discussions in this number deal with questions of analytic geomet y. Professor Borger shows how curves may frequently be plotted advantageously by geometric construction of the points rather than by computation. He illustrates the method by a number of examples, both in rectangular and in polar coördinates, and also in parametric form. It is fairly clear that this method may at times be more useful and at other times much less useful than the usual plan. Surely the student should begin learning as early as possible that the mere laborious plotting of points, either arithmetically or geometrically, is generally to be used only as an incidental aid in determining the form of the curve, the important information being obtained from a functional study of the equation, first by the machinery of algebra, later also by that of the calculus.

Pro'essor Bradshaw gives an interesting exposition of the inaccuracy of the figures usually given in the text-books on solid analytic geometry. The reader who has never before given attention to this matter will scarcely credit, until after direct exam'nation, the uniformity of this error in our usual texts.

I. On Some Geometric Methods for Curve Tracing.[1]

By R. L. Borger, Ohio University.

In beginning courses in analytic geometry, curve tracing is confined almost exclusively to the process of making a table and plotting the points of the curve from the computed coördinates. This destroys in a great measure the geometric aspect of the problem, and at the same time develops in the student a quality of dependence upon the table even in cases where a geometric treatment would be simpler. For many curves in polar form, plotting from a table becomes excessively onerous. If the radius vector is a trigonometric function of the vectorial angle, the interposition of two tables becomes necessary. The student finds too that he has no means of detecting the character of a curve as he has in rectangular coördinates. Such simple curves as

[1] Read before the Mathematical Association of America, Ohio Section, Columbus, April 2, 1920.